MATH 1650: SECTION 5.6: FUNCTION INVERSES

RECALL: One way to think of a function is as a **process**. In this section, we seek a way to reverse that process.

DEFINITION: A function f is called **invertible** if there is a function f^{-1} (read 'f-inverse') such that:

$(f^{-1} \circ f)(x) = x$ for all x in the domain of f **AND** $(f \circ f^{-1})(x) = x$ for all x in the domain of f^{-1}.

NOTE: The equation $(f^{-1} \circ f)(x) = x$ algebraically states that f^{-1} undoes what f did to the input x.

QUESTION: If f is invertible, what is $(f^{-1})^{-1}$?

EXAMPLE: Verify $f(x) = \dfrac{2x-3}{x+1}$ and $g(x) = \dfrac{x+3}{2-x}$ are inverse functions.

To verify f and g are inverses, we need to check two things:

- Verify $(f \circ g)(x) = x$ for all x in the domain of g:

  We first note the domain of g is all real numbers except $x = 2$. So for $x \neq 2$:

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
&= f\left(\frac{x+3}{2-x}\right) \\
&= \frac{2\left(\frac{x+3}{2-x}\right) - 3}{\left(\frac{x+3}{2-x}\right) + 1} \\
&= \frac{2\left(\frac{x+3}{2-x}\right) - 3}{\left(\frac{x+3}{2-x}\right) + 1} \cdot \frac{2-x}{2-x} \\
&= \frac{2(x+3) - 3(2-x)}{(x+3) + (2-x)} \\
&= \frac{2x + 6 - 6 + 3x}{5} \\
&= \frac{5x}{5} \\
(f \circ g)(x) &= x \checkmark
\end{aligned}$$

EXAMPLE: (Continued.) Let $f(x) = \dfrac{2x-3}{x+1}$ and $g(x) = \dfrac{x+3}{2-x}$.

- Verify $(g \circ f)(x) = x$ for all x in the domain of f:

- Use $f^{-1}(x)$ to solve: $\dfrac{2x-3}{x+1} = 6$

  Since $f(x) = \dfrac{2x-3}{x+1}$, we can view the equation $\dfrac{2x-3}{x+1} = 6$ as $f(x) = 6$.

  Since f is invertible, we can apply f^{-1} to both sides of the equation: $f^{-1}\left(f(x)\right) = f^{-1}(6)$ so $x = f^{-1}(6)$.

  Since we have shown $f^{-1} = g$, we have $x = f^{-1}(6) = g(6) = \dfrac{6+3}{2-6} = -\dfrac{9}{4}$.

- Use $g^{-1}(x)$ to solve: $\dfrac{x+3}{2-x} = 1$

ALGEBRAIC PROPERTIES OF INVERSE FUNCTIONS: Suppose f is invertible with inverse f^{-1}:

- the domain of f is the range of f^{-1} and the range of f is the domain of f^{-1}.
- $f(a) = b \iff f^{-1}(b) = a$
- to obtain a formula for $y = f^{-1}(x)$, solve $x = f(y)$ for y.
  i.e., interchange the 'x' and the 'y' in the equation $y = f(x)$.

EXAMPLE: The function $f(x) = \dfrac{2x}{x-1}$ is invertible. Find a formula for $f^{-1}(x)$ and check your answer.

We begin by writing $y = f(x)$ and interchanging x and y:

$$\begin{aligned}
y &= f(x) \\
y &= \frac{2x}{x-1} \\
x &= \frac{2y}{y-1} \\
x(y-1) &= 2y \\
xy - x &= 2y \\
xy - 2y &= x \\
y(x-2) &= x \\
y &= \frac{x}{x-2} \\
f^{-1}(x) &= \frac{x}{x-2}
\end{aligned}$$

We verify: $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f: For all real numbers $x \neq 1$:

$$\begin{aligned}
\left(f^{-1} \circ f\right)(x) &= f^{-1}\left(\frac{2x}{x-1}\right) = \frac{\dfrac{2x}{x-1}}{\dfrac{2x}{x-1} - 2} \\
&= \frac{\dfrac{2x}{x-1}}{\dfrac{2x}{x-1} - 2} \cdot \frac{x-1}{x-1} \\
&= \frac{2x}{2x - 2(x-1)} \\
&= \frac{2x}{2x - 2x + 2} \\
\left(f^{-1} \circ f\right)(x) &= \frac{2x}{2} = x \checkmark
\end{aligned}$$

EXAMPLE: (Continued.) $f(x) = \dfrac{2x}{x-1}$ and $f^{-1}(x) = \dfrac{x}{x-2}$.

Verify: $\left(f \circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}:

EXAMPLE: Consider: $f(x) = \dfrac{2x}{x-1}$ and $f^{-1}(x) = \dfrac{x}{x-2}$

- What is the **vertical** asymptote to the graph of $y = f(x)$?

- What is the **horizontal** asymptote to the graph of $y = f^{-1}(x)$?

- What is the **horizontal** asymptote to the graph of $y = f(x)$?

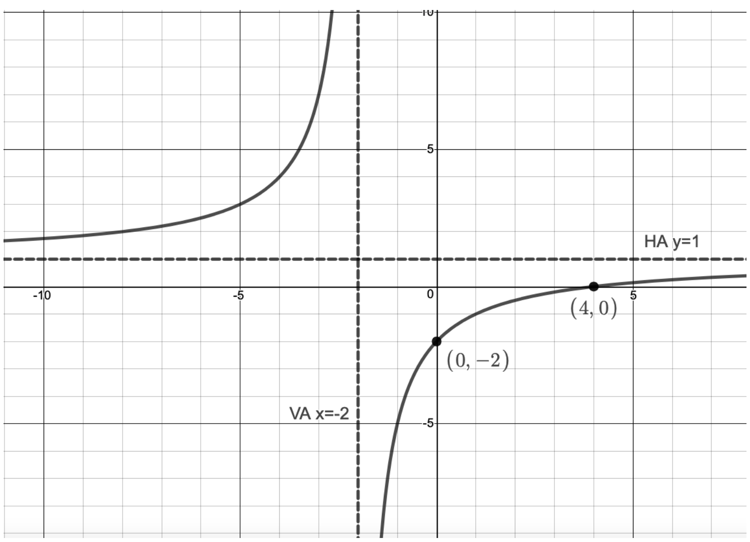
- What is the **vertical** asymptote to the graph of $y = f^{-1}(x)$?

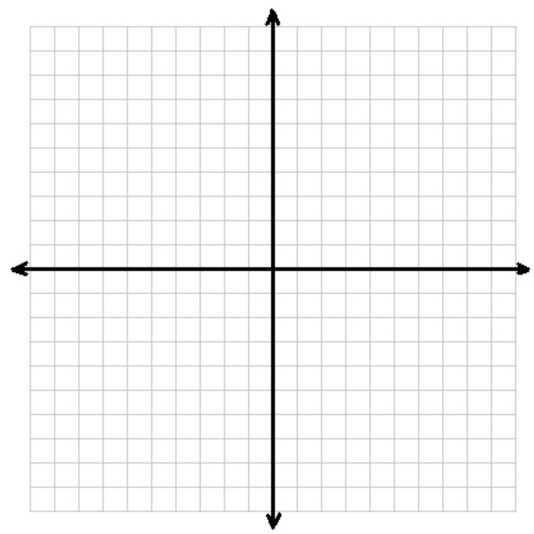
The relationships between the asymptotes of the graphs of f and f^{-1} in the example above are one instance of:

GEOMETRIC PROPERTIES OF INVERSE FUNCTIONS: Suppose f is invertible with inverse f^{-1}:

- (a, b) is on the graph of $y = f(x) \iff (b, a)$ is on the graph of $y = f^{-1}(x)$.
- the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections about the line $y = x$.

EXAMPLE: Use the graph of $y = f(x)$ below on the left to sketch the graph of $y = f^{-1}(x)$ below on the right.

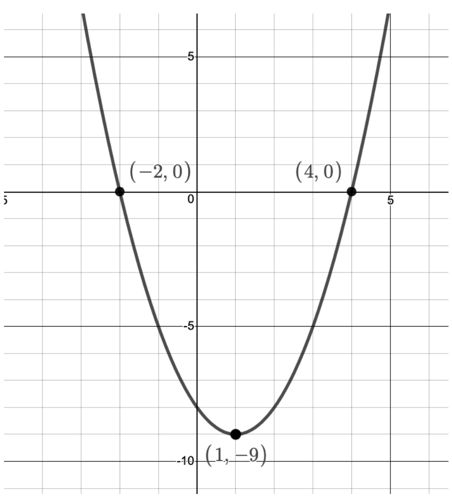


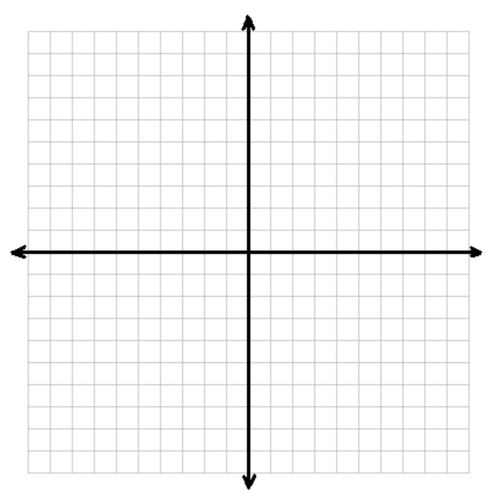


$y = f(x)$

$y = f^{-1}(x)$.

EXAMPLE: Use the graph of $y = f(x)$ on the left to (try to) sketch the graph of $y = f^{-1}(x)$ on the right. What goes wrong?





$y = f(x)$

$y = f^{-1}(x)$?

In the last example, the graph of $y = f(x)$ contained the points $(-2, 0)$ and $(4, 0)$ which meant the graph of $y = f^{-1}(x)$ would have to contain the points $(0, -2)$ and $(0, 4)$, violating the Vertical Line Test.

That is, because two different **inputs** to f (namely -2 and 4) matched with the same **output** from f (namely 0), there was a single **input** to f^{-1} (namely 0) which matched to two different **outputs** (namely -2 and 4.)

Hence, for a function f to have an inverse, each **output** from f comes from only one **input**.

DEFINITION: A function f is said to be **one-to-one** if whenever $f(a) = f(b)$, then $a = b$.
That is, if the outputs from f are the same, the inputs must be the same.

NOTE: To be a **function**, each input can only have one output; to be a **one-to-one** function, each output can only come from one input. So a one-to-one function matches up each input to only one output and vice-versa.

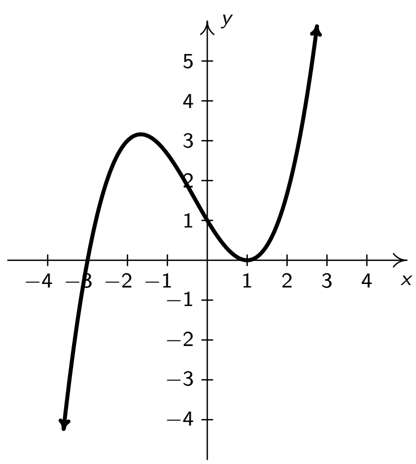
THEOREM: A function is invertible if and only if it is one-to-one. (Do you see why this is true?)

One-to-one functions can be detected geometrically . . .

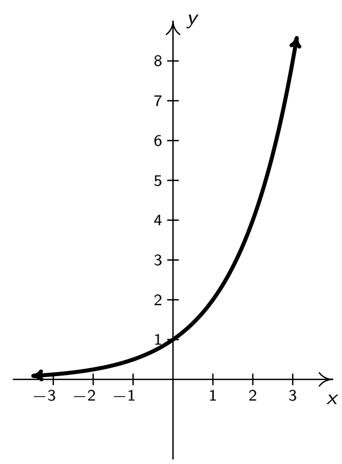
THE HORIZONTAL LINE TEST: A function is one-to-one if and only if every horizontal line intercepts the graph of the function at most once. (Do you see why this works?)

NOTE: Using the Horizontal Line Test on the graph of $y = f(x)$ is equivalent to using the Vertical Line Test on the graph of $y = f^{-1}(x)$. (Do you see why this is true?)

EXAMPLE: Use the Horizontal Line Test to determine which of the functions below have inverses.



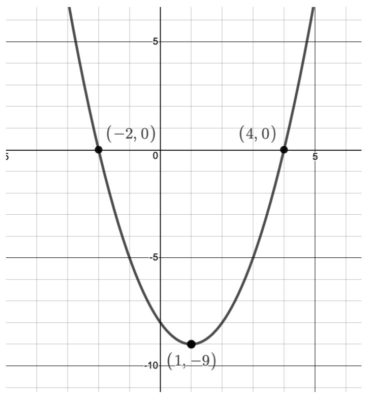
$y = f(x)$



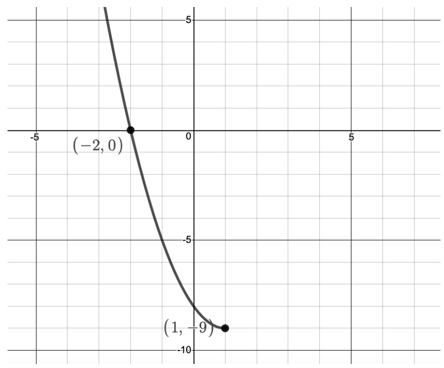
$y = g(x)$

RESTRICTING THE DOMAIN TO FIND (PARTIAL) INVERSES:

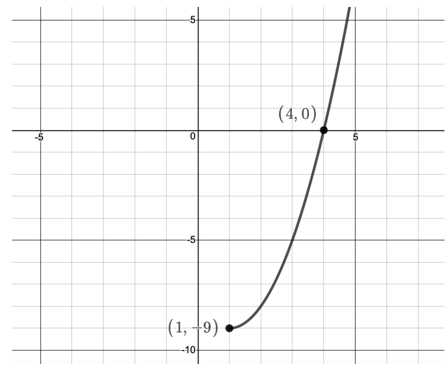
Recall the function f whose graph appears below on the left does **not** have an inverse since it is no one-to-one. (The graph of f fails the HLT.) If we **restrict** our attention to one half of the graph or the other by **restricting** the domain we get the graphs of functions g and h which are one-to-one:



The graph of $y = f(x)$.

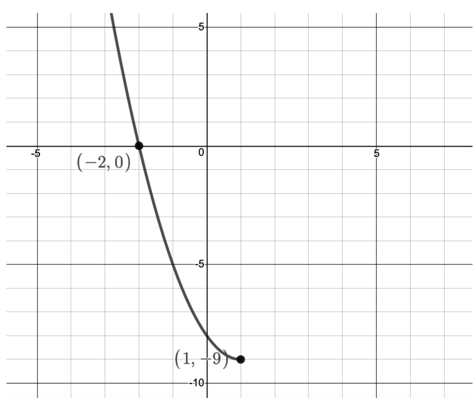


$g(x) = f(x)$ for $x \leq 1$

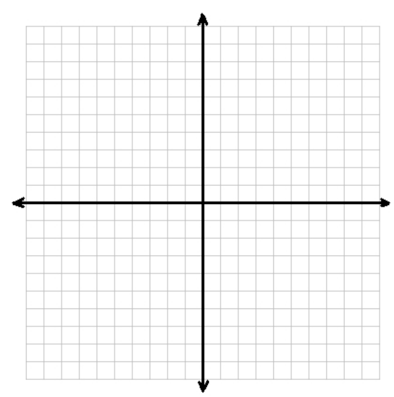


$h(x) = f(x)$ for $x \geq 1$

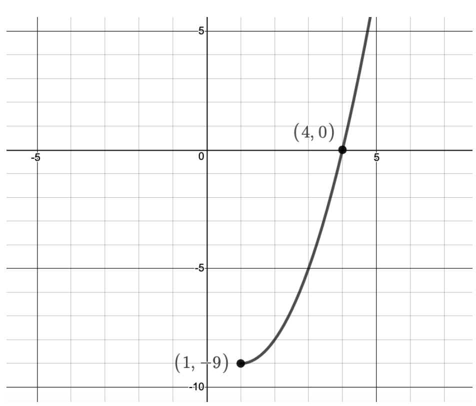
EXAMPLE: Graph the inverses of g and h below.



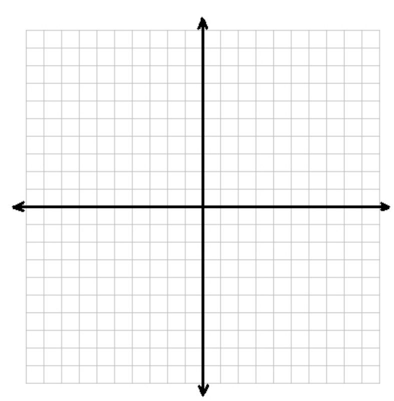
$y = g(x)$



$y = g^{-1}(x)$.



$y = h(x)$



$y = h^{-1}(x)$.

EXAMPLE: Let $f(x) = 10x - x^2$ for $x \leq 5$.

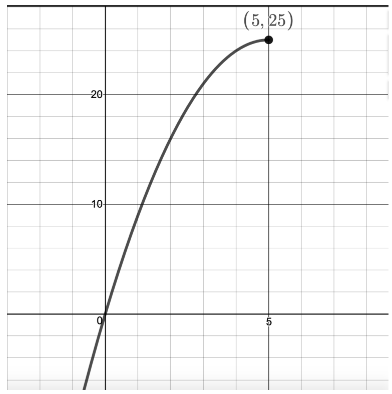
- Graph $y = f(x)$ and use the HLT to show f is one-to-one.

  Rewriting $f(x) = -x^2 + 10x$ for $x \leq 5$, we see f is a quadratic function.

  We find the vertex is at $x = -\dfrac{b}{2a} = -\dfrac{10}{2(-1)} = 5$ and $y = f(5) = -(5)^2 + 10(5) = 25$, so $(5, 25)$.

  Since the domain of f is restricted to $x \leq 5$, we graph only the **left** half of the parabola.

  Since we are only considering half the parabola, we can be assured the graph of f passes the HLT.



- Find a formula for $f^{-1}(x)$.

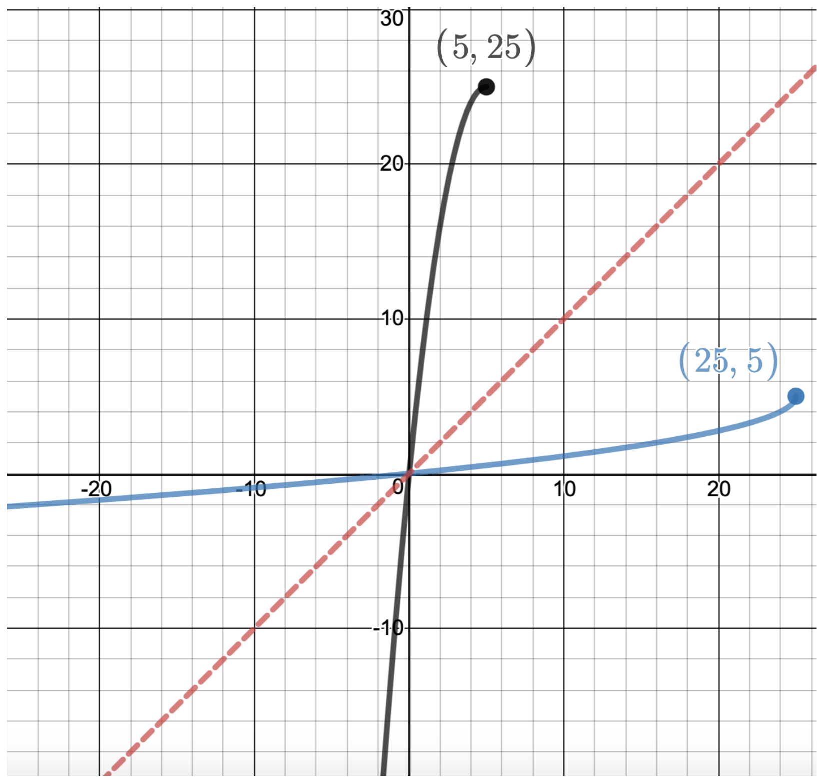
$$
\begin{aligned}
y &= -x^2 + 10x, \quad x \leq 5 && \text{write } y = f(x) \\
x &= -y^2 + 10y, \quad y \leq 5 && \text{interchange } x \text{ and } y \\
y^2 - 10y + x &= 0, \quad y \leq 5 \\
y &= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(x)}}{2(1)}, \quad y \leq 5 && \text{Quadratic Formula: } a = 1,\ b = -10,\ c = x \\
y &= \frac{10 \pm \sqrt{100 - 4x}}{2}, \quad y \leq 5 \\
y &= \frac{10 \pm \sqrt{4(25 - x)}}{2}, \quad y \leq 5 \\
y &= \frac{10 \pm 2\sqrt{25 - x}}{2}, \quad y \leq 5 \\
y &= \frac{2\left(5 \pm \sqrt{25 - x}\right)}{2}, \quad y \leq 5 \\
y &= 5 \pm \sqrt{25 - x}, \quad y \leq 5 \\
y &= 5 - \sqrt{25 - x}, \quad y \leq 5 && \text{since } y \leq 5, \text{ we choose '}-\text{'.} \\
f^{-1}(x) &= 5 - \sqrt{25 - x}
\end{aligned}
$$

- Check your answer algebraically by:
  - simplifying $\left(f^{-1} \circ f\right)(x)$.

$$\begin{aligned}
\left(f^{-1} \circ f\right)(x) &= f^{-1}\left(-x^2 + 10x\right), \quad x \leq 5 \\
&= 5 - \sqrt{25 - (-x^2 + 10x)}, \quad x \leq 5 \\
&= 5 - \sqrt{x^2 - 10x + 25}, \quad x \leq 5 \\
&= 5 - \sqrt{(x-5)^2}, \quad x \leq 5 \\
&= 5 - |x - 5|, \quad x \leq 5 \\
&= 5 - (-(x-5)) \qquad \text{since } x \leq 5,\ |x-5| = -(x-5) \text{ (Section 1.3)} \\
&= 5 + x - 5 \\
\left(f^{-1} \circ f\right)(x) &= x \checkmark
\end{aligned}$$

  - simplifying $\left(f \circ f^{-1}\right)(x)$:

- Geometrically, we have $y = f(x)$ and $y = f^{-1}(x)$ graphed below along with the line of symmetry $y = x$.



HOMEWORK: Section 5.6: 1 - 37 odd